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Theoretical and Numerical Studies of Fractional Volterra-Fredholm Integro-Differential Equations in Banach Space

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Abstract

This paper examines the theoretical, analytical, and approximate solutions of the Caputo fractional Volterra-Fredholm integro-differential equations (FVFIDEs). Utilizing Schaefer's fixed-point theorem, the Banach contraction theorem and the Arzelà-Ascoli theorem, we establish some conditions that guarantee the existence and uniqueness of the solution. Furthermore, the stability of the solution is proved using the Hyers-Ulam stability and Gronwall-Bellman's inequality. Additionally, the Laplace Adomian decomposition method (LADM) is employed to obtain the approximate solutions for both linear and non-linear FVFIDEs. The method's efficiency is demonstrated through some numerical examples.

Keywords: Caputo fractional derivative; Hyers-Ulam stability; Laplace Adomian decomposition method.

1 Introduction

Over the last few decades, the applications of fractional calculus have flourished rapidly in various fields referred to its memory effect's property, and the developments of the newest fractional operators which provide the appropriate environment to modeling the real-life problems such as cancer disease [17], COVID-19 [2], and the transportation of solute in porous media [37]. Fractional integro-differential equations (FIDEs) have gained great attention for modeling some complicated nonlinear phenomenons in a variety of domains, including science, engineering, electromagnetic, fluid-dynamic, traffic, and many more [42]. These equations with their nature, have some complexity in proving the existence, uniqueness, stability of solutions, and finding the exact and approximate solutions. Due to its complexity, many researchers have exclusively focused on both theoretical and numerical aspects of these equations in their works.

Some recent publications have investigated the existence and uniqueness of solutions for various types of FIDEs. For instance, Hussain et al. [21] used the Banach fixed point theorem and semi-group theory to study the existence of mild solution and controllability of semi-linear and non-local FVFIDEs. Raja and Vijayakumar [32] proved the existence of mild solution of inclusion type FVFIDEs with fractional order between 0 and 1 using the fixed point theorem. Laadjal and Ma [26] employed the Banach and Krasnoselskii fixed point theorems to prove the existence and uniqueness of non-linear boundary value Caputo FVFIDEs. Beni [33] presented the existence of the solution of FVFIDEs subject to mixed boundary conditions using the contraction mapping theorem. They also obtained the approximate solutions using Legendre wavelets with the help of the quadrature rule.

Additionally, Amin et al. [6] investigated the existence and uniqueness of non-linear FVFIDEs using the fixed point approach. Verma and Kumar [40] employed the fixed point theorems and the Banach contraction principle to study the existence and uniqueness of the solution of FVFIDEs and solved it using the two-step Adomian decomposition method. Hamoud [18] examined the existence and uniqueness of solutions for neutral types of FVFIDEs using the Leray-Schauder nonlinear alternative and the Krasnoselskii fixed point theorem. Khaldi et al. [24] applied Schauder's fixed-point theorem and the Banach contraction principle to study the existence of solutions for non-linear ψ-Caputo FVFIDEs with non-local initial conditions and analyzed the Hyers-Ulam-Rassias stability of solutions. Moreover, many researchers have investigated the stability of solution of FIDEs. Alam and Shah [3] applied the Hyers-Ulam stability (H.U. stability) to couple the implicit FIDEs, while Shah and Gul [36] studied the different types of stability for the solution of Caputo-Fabrizio FIDEs. Ismaael [22] used the fixed-point theorems to investigate the existence and uniqueness of solution for the impulsive η-Hilfer non-linear FVFIDEs with multi-point fractional boundary non-instantaneous conditions. Karande [23] proved the existence of solution for the functional FIDEs used fixed-point theorem, the generalized Lipschitz, caratheodory and monotonicity conditions.

Different analytical and numerical methods have been employed to approximate the solution of FVFIDEs. Das et al. [12] employed the homotopy perturbation method to find the analytical solutions of FVFIDEs. Ali et al. [4] presented the hybrid combination method using the Bernstein and block-pulse functions wavelet method to approximate the solutions of FVFIDEs with mixed boundary conditions. Hamoud and Ghadle [19] studied the existence and uniqueness of the solution of Caputo FVFIDEs, and their convergence, and approximated the solution using the modified Adomian decomposition method and the modified variational iteration method. Rahimkhani and Ordokhani [31] employed the operational matrix of alternative Legendre functions with the collocation method to approximate the solution of non-linear FVFIDEs with non-local conditions. Amin et al. [7] used the Haar wavelet collocation method to solve FVFIDEs. Furthermore, various methods have been applied to address different types of FIDEs, such as the operational matrices of Müntz-Legendre polynomials [35], block-pulse functions and Fibonacci polynomials [38], operational matrices of the block pulse functions [34], operational matrices of the Lucas wavelets and the Legendre-Gauss quadrature rule [13], the reproducing kernel method [14], the residual power series method [27], the B-spline method [28], Laplace Adomian decomposition method [9]. Other iterative methods can be found in [15, 29].

This paper deals with the initial value problem (IVP) of mixed FVFIDEs of the form:

$$\begin{cases} {}^cD^{\alpha}_{0^+}\mathcal{W}(s) = \varphi(s) + \rho \int_0^s \int_0^C \mathcal{K}(x,v)\mathcal{H}(\omega(v))dvdx, \\ \mathcal{W}(0) = w_0, \quad \mathcal{W}^{'}(0) = w_1, \end{cases} \tag{1}$$

where $\alpha \in (1, 2]$, $0 \leq s, x \leq C$, $\varphi : J = [0, C] \to \mathbb{R}$, is a continuous function and $\mathcal{K}(x, v)$ represents a continuous arbitrary kernel function, a continuous function $\mathcal{H}(\mathcal{W}(s))$ contains both linear and non-linear parts, and $\mathcal{W}(s)$ is an unknown function. The operator ${}^cD^{\alpha}$ denotes Caputo's fractional derivative.

The equation of the form mentioned in (1) has been solved by Wazwaz [41] only with the second integer order using various methods such as the variational iteration method, the series solution method and the direct computation method. However, there has been no study on the same equation in fractional order $1 < \alpha < 2$. Therefore, in this paper, we explore the existence, uniqueness, and stability of solution for the IVP presented in (1). Furthermore, we use the Laplace Adomian decomposition method (LADM) to obtain the approximate solution for the given equation.

This paper is organized as follows: In Section 1, we mention some related works. In Section 2, we provide the basic definitions and theorems related to fractional calculus. Section 3, we introduce the theoretical results, we prove the existence and uniqueness of the solution of mixed FVFIDEs, and then investigate the stability of the solution using the H.U. stability. Section 4 is dedicated to finding the approximate solution by applying the LADM. In Section 5, we present some numerical examples to demonstrate the practical results of the developed theorems. Finally, in Section 6, we summarize our work in the conclusion.

2 Preliminary and Basic Definitions

In this section, we provide some important definitions and theorems in fractional calculus and fixed-point theorem in Banach space.

Let the Banach space $\mathfrak{M} = (J, \mathbb{R})$ denotes all continuous functions on J, such that for any function $\mathcal{W} \in \mathfrak{M}$, the norm $\|\mathcal{W}(s)\|_\infty = \sup\{|\mathcal{W}(s)| : s \in J\}$.

Definition 2.1. [25] *The Riemann-Liouville fractional integral of real order $\alpha > 0$ of a function $\mathcal{W}(s)$ is given by*

$$J^{\alpha}_{a^+}\mathcal{W}(s) = \frac{1}{\Gamma(\alpha)} \int_a^s (s-v)^{\alpha-1}\mathcal{W}(v)dv,$$

where Γ is Euler's Gamma function.

Definition 2.2. [25] *The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $\mathcal{W}(s)$ is defined by*

$$ {}^{c}D^{\alpha}_{a^+}\mathcal{W}(s) = J^{n-\alpha}_{a^+}\left(\frac{d^n}{ds^n}\mathcal{W}(s)\right) = \frac{1}{\Gamma(n-\alpha)}\int_a^s (s-v)^{n-\alpha-1}\mathcal{W}^{(n)}(v)dv, \quad \alpha > 0, $$

where $n = [\alpha] + 1$ and Γ represents Gamma function.

The following properties are well known in fractional calculus [25]. Let $\alpha > 0$ and $\beta > 0$, and $\mathcal{W} \in L^1[a, b]$. Then,

$$ J^{\alpha}_{a^+}J^{\beta}_{a^+}\mathcal{W}(s) = J^{\beta}_{a^+}J^{\alpha}_{a^+}\mathcal{W}(s) = J^{\alpha+\beta}_{a^+}\mathcal{W}(s), \tag{2} $$

$$ {}^{c}D^{\alpha}_{a^+}\left[J^{\alpha}_{a^+}\mathcal{W}(s)\right] = \mathcal{W}(s), \tag{3} $$

$$ J^{\alpha}_{a^+}\left[{}^{c}D^{\alpha}_{a^+}\mathcal{W}(s)\right] = \mathcal{W}(s) - \sum_{k=0}^{n-1}\frac{\mathcal{W}^{(k)}(a)}{k!}(s-a)^k, \quad \text{for} \quad n-1 < \alpha \le n. \tag{4} $$

Also, the fractional integral acts on a power function according to the following formula:

$$ J^{\beta}_{a^+}(s-a)^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\beta+\mu+1)}(s-a)^{\beta+\mu}, \tag{5} $$

where $\mu > -1$.

Theorem 2.1. [5] *If $\mathcal{U}(s)$ and $\mathcal{V}(s)$ are piece-wise continuous functions on $[0, \infty]$. Then, the Laplace transform of the convolution of both functions is given by*

$$ \mathcal{L}\Big((\mathcal{U} * \mathcal{V})(s)\Big) = \mathcal{L}(\mathcal{U}(s))\mathcal{L}(\mathcal{V}(s)). \tag{6} $$

Lemma 2.1. [5] *If $W(\tau)$ is the Laplace transform of a function $\mathcal{W}(s)$, then for any $\delta > 0$, the Laplace transform of the Caputo fractional derivative ${}^{c}_{a}\mathcal{D}^{\delta}\mathcal{W}(s)$ is given by*

$$ \mathcal{L}\left[{}^{c}_{a}\mathcal{D}^{\delta}\mathcal{W}(s)\right] = \tau^{\delta}W(\tau) - \sum_{j=0}^{m-1}\tau^{\delta-j-1}\mathcal{W}^{(j)}(0), \tag{7} $$

where $m = [\delta] + 1$.

Theorem 2.2. [43] [*The Banach contraction principle*] *Let (X, d) be a complete metric space, and $\mathcal{W} : X \to X$ a contraction mapping ($d(\mathcal{W}x, \mathcal{W}y) \le kd(x, y)$, where $0 < k < 1$, for each $x, y \in X$). Then, there exists a unique fixed point s of $\mathcal{W}$ in X, i.e., $\mathcal{W}s = s$.*

Theorem 2.3. [30, 10] [*The Schaefer fixed point theorem*] *Suppose that $\mathfrak{M} = (X, \mathbb{R})$ is a Banach space. Let $\Omega : \mathfrak{M} \to \mathfrak{M}$ be a completely, continuous mapping. If the set $\{s \in X : s = \lambda\Omega s$ for some $\lambda \in (0, 1)\}$ is bounded, then Ω has a fixed point.*

Theorem 2.4. [43] [*Arzelà-Ascoli theorem*] *If a family $\mathcal{W} = \{\mathcal{W}(s)\}$ in $C(J, \mathbb{R})$ is uniformly bounded and equicontinuous on J, and for any $s^* \in J$, $\{\mathcal{W}(s^*)\}$ is relatively compact, then $\mathcal{W}$ has a uniformly convergent subsequence $\{\mathcal{W}_n(s)\}_{n=1}^{\infty}$.*

Arzelà-Ascoli theorem is the key to the following result [43]: "A subset $\mathcal{W}$ in $C(J,\mathbb{R})$ is relatively compact if and only if it is uniformly bounded and equicontinuous on J".

Lemma 2.2. [1] [*Gronwall-Bellman's inequality*] *Let $\mathcal{W}(s)$ and $h(s)$ be non-negative continuous functions defined on $I=[\alpha,\alpha+h]$ and c be a non-negative constant. If*

$$\mathcal{W}(s) \leqslant c + \int_{\alpha}^{s} h(v)\mathcal{W}(v)dv, \quad s \in I, \tag{8}$$

then

$$\mathcal{W}(s) \leqslant c * \exp\left(\int_{\alpha}^{s} h(v)dv\right), \quad s \in I. \tag{9}$$

3 Theoretical Results

In this section, we present some results related to the existence, uniqueness, and stability of solution of IVP (1).

3.1 Existence and uniqueness

We suggest the following axioms:

(A1) Let $\mathcal{H}$ be a continuous function. There exists a constant $L>0$, for each $x,y\in\mathbb{R}$, such that

$$\|\mathcal{H}(x)-\mathcal{H}(y)\| \leq L\|x-y\|.$$

(A2) Let $\mathcal{K}(x,v)$ be a continuous arbitrary kernel function satisfies

$$\mathcal{K}^* = \sup_{x,v\in J} \mathcal{K}(x,v) = \|\mathcal{K}(x,v)\|,$$

and $\varphi(s)$ be a continuous bounded function, such that $\sup_{s\in J}|\varphi(s)| = \|\varphi(s)\|$.

(A3) Let $\mathcal{H}$ be a continuous and bounded function on a normed vector space, such that there exists a constant $z>0$, and for each $s\in J$, $\mathcal{W}\in\mathbb{R}$, $\|\mathcal{H}(\mathcal{W}(s))\| \leq z\|\mathcal{W}\|$.

Lemma 3.1. *Let $1<\alpha\leq 2$, and suppose that $\mathcal{H},\mathcal{K}$ and φ are arbitrary continuous functions defined on $\mathfrak{M}=(J,\mathbb{R})$. If $\mathcal{W}\in\mathfrak{M}=(J,\mathbb{R})$, and $s\in J$, then $\mathcal{W}$ satisfies the IVP (1) if and only if $\mathcal{W}$ satisfies the integral equation,*

$$\begin{aligned}\mathcal{W}(s) = w_0 + w_1 s + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\omega)^{\alpha-1}\varphi(\omega)d\omega \\ + \frac{\rho}{\Gamma(\alpha)}\int_0^s (s-\omega)^{\alpha-1}\int_0^{\omega}\int_0^{C} \mathcal{K}(x,r)\mathcal{H}(\mathcal{W}(r))drdxd\omega.\end{aligned} \tag{10}$$

Proof. Applying the Riemann-Liouville fractional integral of order α to Equation (1) gives

$$J^{\alpha}_{0+}\left({}^cD^{\alpha}_{0+}\mathcal{W}(s)\right) = J^{\alpha}_{0+}(\varphi(s)) + \rho J^{\alpha}_{0+}\left(\int_0^s\int_0^C \mathcal{K}(x,v)\mathcal{H}(\mathcal{W}(v))dvdx\right). \tag{11}$$

Using Equation (4) in Equation (11) with $a = 0$ and $n = 1$, we have

$$\mathcal{W}(s) - \mathcal{W}^{(0)}(0) - \mathcal{W}^{(1)}(0)s = J^{\alpha}_{0+}(\varphi(s)) + \rho J^{\alpha}_{0+}\left(\int_0^s \int_0^C \mathcal{K}(x,v)\mathcal{H}(\mathcal{W}(v))dvdx\right). \tag{12}$$

The result follows by substituting the initial conditions in Equation (12), thus completes the proof. □

To prove our main results for the existence of solution, we convert the IVP (1) to the fixed point problem by defining the operator $\Omega_{\mathcal{W}} : \mathfrak{M} = (J = [0, C], \mathbb{R}) \to \mathfrak{M} = (J, \mathbb{R})$ as

$$\begin{aligned}\Omega_{\mathcal{W}}(s) &= w_0 + w_1 s + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\omega)^{\alpha-1}\varphi(\omega)d\omega \\ &\quad + \frac{\rho}{\Gamma(\alpha)}\int_0^s (s-\omega)^{\alpha-1}\int_0^{\omega}\int_0^C \mathcal{K}(x,r)\mathcal{H}(\mathcal{W}(r))drdxd\omega. \end{aligned} \tag{13}$$

Then, using the Schaefer fixed point theorem and Arzelà-Ascoli theorem in Theorems 2.3 and 2.4, respectively, we demonstrate the existence of solution in the following theorem.

Theorem 3.1. *Let $\mathcal{H}, \mathcal{K}$ and φ be arbitrary continuous functions on $C(J, \mathbb{R})$ and satisfy the Axioms (A1) - (A3). Let $\Omega_{\mathcal{W}} : \mathfrak{M} \to \mathfrak{M}$ be the operator defined in Equation (13). Then, the IVP (1) has at least one solution on J.*

Proof. The operator $\Omega_{\mathcal{W}}$ is completely continuous and bounded are proved as follows:

Step 1: The operator $\Omega_{\mathcal{W}}$ is continuous. Let $\{\mathcal{W}_n\}$ be a sequence such that $\mathcal{W}_n \to \mathcal{W}$ in $\mathfrak{M}$. Then, for each $s \in J$, we have

$$\begin{aligned}&\|\Omega_{\mathcal{W}_n}(s) - \Omega_{\mathcal{W}}(s)\| \\ &\leq \frac{\rho}{\Gamma(\alpha)}\int_0^s (s-\omega)^{\alpha-1}\int_0^{\omega}\int_0^C |\mathcal{K}(x,r)|\|\mathcal{H}(\mathcal{W}_n(r)) - \mathcal{H}(\mathcal{W}(r))\|drdxd\omega \\ &\leq \frac{\rho L}{\Gamma(\alpha)}\int_0^s (s-\omega)^{\alpha-1}\int_0^{\omega}\int_0^C \sup_{s\in J}|\mathcal{K}(x,r)|\,\|\mathcal{W}_n(s) - \mathcal{W}(s)\|\,drdxd\omega \\ &\leq \mathcal{R}\,\|\mathcal{W}_n(s) - \mathcal{W}(S)\|\,,\end{aligned}$$

where,

$$\mathcal{R} := \frac{\rho L \mathcal{K}^* C^{\alpha+2}}{\Gamma(\alpha+2)}. \tag{14}$$

By the continuity of $\mathcal{H}$ and $\mathcal{K}$, and $\mathcal{W}_n \to \mathcal{W}$, we have

$$\|\Omega_{\mathcal{W}_n}(s) - \Omega_{\mathcal{W}}(s)\|_{\infty} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

Step 2: The operator $\Omega_{\mathcal{W}}$ is bounded on $\mathfrak{M}$. It is sufficient to demonstrate that for

$$\varepsilon > 0, \quad \exists\, Q > 0, \quad \ni \forall\, \mathcal{W} \in Nb(\varepsilon) := B_{\varepsilon} = \{\mathcal{W} \in \mathfrak{M} = (J, \mathbb{R}) : \|\mathcal{W}\|_{\infty} \leq \varepsilon\},$$

we have $\|\Omega_{\mathcal{W}}(s)\|_\infty \leq Q$. Thus, $\forall\ s \in J$, we have

$$\begin{aligned}
&\|\Omega_{\mathcal{W}}(s)\|_\infty \\
&\quad \leq |w_0 + w_1 s| + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \sup_{\omega \in J} |\varphi(\omega)| d\omega + \frac{\rho}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \\
&\qquad \times \int_0^\omega \int_0^C \sup_{x,r\in J} |\mathcal{K}(x,r)| \sup_{r \in J} |\mathcal{H}(\mathcal{W}(r))| dr dx d\omega \\
&\quad \leq \|w_0\| + \|w_1\|\, C + \frac{C^\alpha}{\alpha\Gamma(\alpha)} \|\varphi(s)\| + \frac{\rho C^{\alpha+2}\mathcal{K}^*}{\alpha(\alpha+1)\Gamma(\alpha)} \|\mathcal{H}(\mathcal{W}(s))\|.
\end{aligned}$$

By using Axioms (A3), we have

$$\begin{aligned}
\|\Omega_{\mathcal{W}}(s)\|_\infty &\leq \|w_0\| + \|w_1\|\, C + \frac{C^\alpha}{\Gamma(\alpha+1)} \|\varphi(s)\| + \frac{\rho C^{\alpha+2}\mathcal{K}^*}{\Gamma(\alpha+2)} z \|\mathcal{W}\| \\
&\leq \|w_0\| + \|w_1\|\, C + \frac{C^\alpha}{\Gamma(\alpha+1)} \|\varphi(s)\| + \frac{\rho C^{\alpha+2}\mathcal{K}^*}{\Gamma(\alpha+2)}\, \mathrm{z}\, \varepsilon := \mathrm{Q}.
\end{aligned} \tag{15}$$

Step 3: The operator $\Omega_{\mathcal{W}}$ is equicontinuous. Let $s_1, s_2 \in J$ be arbitrary, such that $s_1 < s_2$. Then,

$$\begin{aligned}
&\|\Omega_{\mathcal{W}}(s_2) - \Omega_{\mathcal{W}}(s_1)\| \\
&\quad \leq \|w_1(s_2 - s_1)\| + \frac{1}{\Gamma(\alpha)} \left(\int_0^{s_2} (s_2-\omega)^{\alpha-1} - \int_0^{s_1} (s_1-\omega)^{\alpha-1} \right) \|\varphi(s)\| d\omega \\
&\qquad + \frac{\rho}{\Gamma(\alpha)} \left(\int_0^{s_2} (s_2-\omega)^{\alpha-1} - \int_0^{s_1} (s_1-\omega)^{\alpha-1} \right) \\
&\qquad \times \int_0^\omega \int_0^C \|\mathcal{K}(x,r)\| \|\mathcal{H}(\mathcal{W}(r))\| dr dx d\omega.
\end{aligned}$$

Adding $\pm \int_0^{s_1} (s_2 - s)^{\alpha-1}\, ds$ for each term inside the parenthesis, and after simplification, we have

$$\begin{aligned}
&\|\Omega_{\mathcal{W}}(s_2) - \Omega_{\mathcal{W}}(s_1)\| \\
&\quad \leq \|w_1(s_2 - s_1)\| + \frac{1}{\Gamma(\alpha)} \bigg(\int_{s_1}^{s_2} (s_2-\omega)^{\alpha-1} + \int_0^{s_1} (s_2-\omega)^{\alpha-1} \\
&\qquad - (s_1-\omega)^{\alpha-1} \bigg) \|\varphi(w)\| dw + \frac{\rho}{\Gamma(\alpha)} \bigg(\int_{s_1}^{s_2} (s_2-\omega)^{\alpha-1} + \int_0^{s_1} (s_2-\omega)^{\alpha-1} \\
&\qquad - (s_1-\omega)^{\alpha-1} \bigg) \times \int_0^\omega \int_0^C \|\mathcal{K}(x,r)\| \|\mathcal{H}(\mathcal{W}(r))\| dr dx d\omega \\
&\quad \leq \|w_1\| (s_2 - s_1) + \frac{(s_2^\alpha - s_1^\alpha)}{\Gamma(\alpha+1)} \|\varphi(s)\| + \frac{\left(s_2^{\alpha+1} - s_1^{\alpha+1}\right) \rho C \mathcal{K}^*}{\Gamma(\alpha+2)}\, \mathrm{z}\, \varepsilon.
\end{aligned}$$

Hence, $\|\Omega_{\mathcal{W}}(s_2) - \Omega_{\mathcal{W}}(s_1)\| \to 0$ as $s_2 \longrightarrow s_1$.

Step 4: To show that $\mho = \{\mathcal{W} \in \mathfrak{M} : \mathcal{W} = \delta\Omega_{\mathcal{W}}\}$ is a bounded set in $\mathfrak{M}$. Let $\mathcal{W} \in \mho$, then $\mathcal{W} = \delta\Omega_{\mathcal{W}}$, for some $\delta \in (0,1)$. Thus, for each $s \in J$, we have

$$\begin{aligned}
\mathcal{W}(s) &= \delta(w_0 + w_1 s) + \frac{\delta}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \varphi(\omega) d\omega \\
&\quad + \frac{\rho\delta}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \int_0^\omega \int_0^C \mathcal{K}(x,r) \mathcal{H}(\mathcal{W}(r)) dr dx d\omega.
\end{aligned}$$

By using the boundedness property of $\|\Omega_{\mathcal{W}}\|$, with $0 < \delta < 1$, and taking the norm to both sides, we have

$$\|\mathcal{W}(s)\| = \|\delta\Omega_{\mathcal{W}}\| = \|\delta\|\|\Omega_{\mathcal{W}}\|\| < \|\Omega_{\mathcal{W}}\| \leq \mathrm{Q},$$

where Q is defined in Equation (15). Thus, $\|\mathcal{W}(s)\| \leq Q$, which proves that $\mho$ is bounded.

From Steps 1-3, the operator $\Omega_{\mathcal{W}}$ is continuous, bounded, and equicontinuous, hence by Theorem 2.4, we deduce that $\Omega_{\mathcal{W}} : \mathfrak{M} \to \mathfrak{M}$ is a completely continuous operator. Together with Step 4, and Theorem 2.3, we conclude that $\Omega_{\mathcal{W}}$ has a fixed point which is a solution of the IVP (1). □

Theorem 3.2. *Let $\mathcal{H}, \mathcal{K}$ and φ be arbitrary continuous functions on $\mathfrak{M}$ and satisfy the Axioms (A1) - (A3), and $\Omega_{\mathcal{W}} : \mathfrak{M} \to \mathfrak{M}$ is a completely continuous bounded operator defined in Equation (13). Furthermore, suppose that there exists a constant $\mathcal{R} := \left(\dfrac{\rho L\mathcal{K}^* C^{\alpha+2}}{\Gamma(\alpha+2)}\right) < 1$, then the IVP (1) has a unique solution on J.*

Proof. For $s \in J$ and $\mathcal{U}, \mathcal{V} \in \mathfrak{M}$, we have

$$\begin{aligned}
\|\Omega_{\mathcal{U}}(s) - \Omega_{\mathcal{V}}(s)\| &\leq \frac{\rho}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \left\| \int_0^\omega \int_0^C \mathcal{K}(x,r)[\mathcal{H}(\mathcal{U}(r)) - \mathcal{H}(\mathcal{V}(r))] dr dx \right\| d\omega \\
&\leq \frac{\rho L}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \int_0^\omega \int_0^C \|\mathcal{K}(x,r)\| \|\mathcal{U}(r) - \mathcal{V}(r)\| dr dx d\omega \\
&\leq \frac{\rho L\mathcal{K}^*}{\Gamma(\alpha)} \left(\int_0^s (s-\omega)^{\alpha-1} \int_0^\omega \int_0^C dr dx d\omega \right) \|\mathcal{U}(s) - \mathcal{V}(s)\| \\
&\leq \frac{\rho L\mathcal{K}^* C^{\alpha+2}}{\Gamma(\alpha+2)} \|\mathcal{U}(s) - \mathcal{V}(s)\| \\
&\leq \mathcal{R}\|\mathcal{U}(s) - \mathcal{V}(s)\|.
\end{aligned}$$

Since $\mathcal{R} < 1$, we conclude that $\Omega_{\mathcal{W}}$ is a contraction. By the Banach contraction theorem (Theorem 2.2), we deduce that Ω has a unique fixed point which is a solution to the IVP (1). □

3.2 Stability of solution

This section investigates the H.U. stability for the solution of the IVP (1).

Definition 3.1. [8] *Let $\mathcal{H}, \mathcal{K}$ and φ be arbitrary continuous functions on $\mathfrak{M}$ and satisfy the Axioms (A1) - (A3). The IVP (1) is H.U. stability, if there exist $\zeta_{\mathcal{H},\mathcal{K},\varphi} \in \mathbb{R} : \zeta_{\mathcal{H},\mathcal{K},\varphi} > 0$, such that for any $\epsilon > 0$ and for any solution $\mathcal{V} \in \mathfrak{M}$ of the inequality*

$$\begin{cases} \left| {}^cD_{0+}^{\alpha}\mathcal{V}(s) - \varphi(s) - \rho \displaystyle\int_0^s \int_0^C \mathcal{K}(x,v)\mathcal{H}(\mathcal{V}(v)) dv dx \right| \leqslant \epsilon, \quad s \in J, \\ \mathcal{V}(0) = v_0, \quad \mathcal{V}'(0) = v_1, \end{cases} \tag{16}$$

there exist $\mathcal{W} \in \mathfrak{M}$, a unique solution of the IVP (1) with

$$\|\mathcal{V}(s) - \mathcal{W}(s)\| \leq \zeta_{\mathcal{H},\mathcal{K},\varphi}\varepsilon. \tag{17}$$

Theorem 3.3. *[H.U. stability] Let $\mathcal{H}, \mathcal{K}$ and φ be arbitrary continuous functions on $\mathfrak{M}$ and satisfy the Axioms (A1) - (A3). If there exist a positive constant ζ, such that*

$$\zeta := \zeta_{\mathcal{H},\mathcal{K},\varphi} := \frac{C^{\alpha}}{\Gamma(\alpha+1)} \times \exp\left(\frac{\rho\, L\mathcal{K}^{*} C^{\alpha+2}}{\Gamma(\alpha+1)}\right),$$

then the IVP (1) is H.U. stability.

Proof. Let $\epsilon > 0$ and let $\mathcal{W} \in C(J, \mathbb{R})$ be a unique solution satisfying Equation (1) with initial conditions $w_0 = \mathcal{V}(0) = v_0, w_1 = \mathcal{V}'(0) = v_1$. By Lemma 3.1, $\mathcal{W}(s)$ satisfies

$$\begin{aligned} \mathcal{W}(s) = w_0 + w_1 s + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \varphi(\omega) d\omega + \frac{\rho}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \\ \times \int_0^{\omega} \int_0^{C} \mathcal{K}(x,r) \mathcal{H}(\mathcal{W}(r)) dr dx d\omega. \end{aligned} \tag{18}$$

Let $\mathcal{V} \in \mathfrak{M}$ be a function satisfies Equation (16). Integrating the inequality in Equation (16) from 0 to s gives

$$\begin{aligned} \Bigg| \mathcal{V}(s) - v(0) - v'(0)s - \frac{1}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \varphi(\omega) d\omega \\ - \frac{\rho}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \int_0^{\omega} \int_0^{C} \mathcal{K}(x,r) \mathcal{H}(\mathcal{V}(r)) dr dx d\omega \Bigg| \le \frac{\epsilon\, s^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned} \tag{19}$$

Using the initial conditions yields

$$\begin{aligned} \Bigg| \mathcal{V}(s) - w_0 - w_1 s - \frac{1}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \varphi(\omega) d\omega \\ - \frac{\rho}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \int_0^{\omega} \int_0^{C} \mathcal{K}(x,r) \mathcal{H}(\mathcal{V}(r)) dr dx d\omega \Bigg| \le \frac{\epsilon\, s^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned} \tag{20}$$

To complete our proof, using the Definition 3.1 and Equation (18), $|\mathcal{V}(s) - \mathcal{W}(s)|$ can be computed as

$$\begin{aligned} |\mathcal{V}(s) - \mathcal{W}(s)| = \Bigg| \mathcal{V}(s) - w_0 - w_1 s - \frac{1}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \varphi(\omega) d\omega - \frac{\rho}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \\ \times \int_0^{\omega} \int_0^{C} \mathcal{K}(x,r) \mathcal{H}(\mathcal{W}(r)) dr dx d\omega \Bigg|. \end{aligned}$$

Adding the terms

$$\left(\pm \frac{\rho}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \int_0^{\omega} \int_0^{C} \mathcal{K}(x,r) \mathcal{H}(\mathcal{V}(r)) dr dx d\omega \right), \tag{21}$$

and simplify

$$\begin{aligned} |\mathcal{V}(s) - \mathcal{W}(s)| &= \left| \mathcal{V}(s) \pm \frac{\rho}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \int_0^{\omega} \int_0^{C} \mathcal{K}(x,r) \mathcal{H}(\mathcal{V}(r)) dr dx d\omega - \mathcal{W}(s) \right| \\ &\le \Bigg| \mathcal{V}(s) - w_0 - w_1 s - \frac{1}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \varphi(\omega) d\omega \\ &\quad - \frac{\rho}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \int_0^{\omega} \int_0^{C} \mathcal{K}(x,r) \mathcal{H}(\mathcal{V}(r)) dr dx d\omega \Bigg| \\ &\quad + \frac{\rho}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \int_0^{\omega} \int_0^{C} \Big| \mathcal{K}(x,r) \left[\mathcal{H}(\mathcal{V}(r)) - \mathcal{H}(\mathcal{W}(r)) \right] \Big| dr dx d\omega. \end{aligned} \tag{22}$$

Using the inequality (20) into Equation (22) gives

$$\begin{aligned} &|\mathcal{V}(s) - \mathcal{W}(s)| \\ &\quad \leq \frac{\epsilon\, s^{\alpha}}{\Gamma(\alpha+1)} + \frac{\rho}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \int_0^{\omega} \int_0^{C} |\mathcal{K}(x,r)| |\mathcal{H}(\mathcal{V}(r)) - \mathcal{H}(\mathcal{W}(r))| dr dx d\omega. \end{aligned} \tag{23}$$

Taking the supremum on both sides of Equation (23) and using Axioms (A1) and (A2) gives

$$\| \mathcal{V}(s) - \mathcal{W}(s) \| \leq \frac{\epsilon\, C^{\alpha}}{\Gamma(\alpha+1)} + \int_0^s \frac{\rho\, L\mathcal{K}^* \mathrm{C}^2}{\Gamma(\alpha)} (C-\omega)^{\alpha-1} \|\mathcal{V}(s) - \mathcal{W}(s)\| d\omega. \tag{24}$$

Applying Gronwall-Bellman's inequality, Lemma 2.2, to the inequality (24) and simplify, we have

$$\begin{aligned} \| \mathcal{V}(s) - \mathcal{W}(s) \| &\leq \frac{\epsilon\, C^{\alpha}}{\Gamma(\alpha+1)} \times \exp\left(\int_0^s \frac{\rho\, L\mathcal{K}^*\, \mathrm{C}^2}{\Gamma(\alpha)} (C-\omega)^{\alpha-1} d\omega \right) \\ &\leq \frac{\epsilon\, C^{\alpha}}{\Gamma(\alpha+1)} \times \exp\left(\frac{\rho\, L\mathcal{K}^*\, \mathrm{C}^{\alpha+2}}{\Gamma(\alpha+1)} \right) \leq \epsilon\, \zeta. \end{aligned}$$

Thus, from Definition 3.1, we conclude that the IVP (1) is H.U. stability. □

Next, we prove the stability of solution of IVP (1) using the definition of H.U. stability (Definition 3.1). We need the following remark.

Remark 3.1. [8] *A function $\mathcal{V} \in \mathfrak{M}$ is a solution of inequalities (16) if and only if there exist a function $\psi \in \mathfrak{M}$ such that*

1. $|\psi(s)| \leq \varepsilon, \quad s \in J,$
2. ${}^{c}D_{0+}^{\alpha} \mathcal{V}(s) = \left(\varphi(s) + \rho \int_0^s \int_0^C \mathcal{K}(x,v) \mathcal{H}(\mathcal{V}(v)) dv dx \right) + \psi(s),$
3. $\mathcal{V}(0) = v_0, \quad \mathcal{V}'(0) = v_1.$

In views of Lemma 3.1, we obtain the following lemma.

Lemma 3.2. *Let $1 < \alpha \leq 2$, and suppose that the functions $\mathcal{H}, \mathcal{K}, \varphi \in \mathfrak{M}$ satisfy the Axioms (A1) - (A3). Let $\psi \in \mathfrak{M}$ satisfies the conditions in Remark 3.1. Then, the following Caputo FIDEs*

$$\begin{cases} {}^{c}D_{0+}^{\alpha} \mathcal{V}(s) = \left(\varphi(s) + \rho \int_0^s \int_0^C \mathcal{K}(x,v) \mathcal{H}(\mathcal{V}(v)) dv dx \right) + \psi(s), \\ \mathcal{V}(0) = v_0 = w_0 + \varepsilon, \quad \mathcal{V}'(0) = v_1 = w_1 + \varepsilon, \end{cases} \tag{25}$$

is equivalent to the integral equation

$$\begin{aligned} \mathcal{V}(s) = v_0 + v_1 s &+ \frac{1}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \varphi(\omega) d\omega \\ &+ \frac{\rho}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \int_0^{\omega} \int_0^{C} \mathcal{K}(x,r) \mathcal{H}(\mathcal{V}(r)) dr dx d\omega \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^s (s-\omega)^{\alpha-1} \psi(\omega) d\omega. \end{aligned} \tag{26}$$

Proof. Applying the Riemann-Liouville fractional integral of order α to Equation (25) gives

$$J_{0+}^{\alpha}({}^{c}D_{0+}^{\alpha}\mathcal{V}(s)) = J_{0+}^{\alpha}(\varphi(s)) + \rho J_{0+}^{\alpha}\left(\int_0^s\int_0^C \mathcal{K}(x,v)\mathcal{H}(\mathcal{V}(v))dvdx\right) + J_{0+}^{\alpha}(\psi(s)). \tag{27}$$

Using Equation (4) in Equation (27) with $a = 0$ and $n = 1$, we have

$$\begin{aligned}\mathcal{V}(s) - \mathcal{V}^{(0)}(0) - \mathcal{V}^{(1)}(0)s = J_{0+}^{\alpha}(\varphi(s)) + \rho J_{0+}^{\alpha}\left(\int_0^s\int_0^C \mathcal{K}(x,v)\mathcal{H}(\mathcal{W}(v))dvdx\right) \\ + J_{0+}^{\alpha}(\psi(s)).\end{aligned} \tag{28}$$

Since $\psi \in \mathfrak{M}$ satisfies the conditions in Remark 3.1, substitute the initial conditions $\mathcal{V}(0), \mathcal{V}'(0)$ into Equation (28), we obtain Equation (26), and hence the proof completes. □

Theorem 3.4. [*H.U. stability*] *Assume that the functions* $\mathcal{H}, \mathcal{K}, \varphi \in \mathfrak{M}$ *satisfy the Axioms (A1) - (A3). If* $\zeta_{\mathcal{H},\mathcal{K},\varphi} = \frac{1}{1-\mathcal{R}}\left(1 + C + \frac{C^{\alpha}}{\Gamma(\alpha+1)}\right) > 0$, *where* $\mathcal{R}$ *is given in Equation (14), then the IVP (1) is H.U. stability.*

Proof. Let $\mathcal{W}(s)$ be a solution of IVP (1) and $\mathcal{V}(s)$ be a solution of inequality (16). By using Lemma 3.2 and Remark 3.1, for any $s \in J$,

$$\begin{aligned}&|\mathcal{V}(s) - \mathcal{W}(s)| \\ &= \Bigg|v_0 + v_1 s + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\omega)^{\alpha-1}\varphi(\omega)d\omega + \frac{\rho}{\Gamma(\alpha)}\int_0^s (s-\omega)^{\alpha-1}\int_0^{\omega}\int_0^C \mathcal{K}(x,r)\mathcal{H}(\mathcal{V}(r))drdxd\omega \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\omega)^{\alpha-1}\psi(\omega)d\omega - w_0 - w_1 s - \frac{1}{\Gamma(\alpha)}\int_0^s (s-\omega)^{\alpha-1}\varphi(\omega)d\omega \\ &\quad - \frac{\rho}{\Gamma(\alpha)}\int_0^s (s-\omega)^{\alpha-1}\int_0^{\omega}\int_0^C \mathcal{K}(x,r)\mathcal{H}(\mathcal{W}(r))drdxd\omega\Bigg| \\ &\le |v_0 - w_0| + |v_1 - w_1|s + \frac{\rho}{\Gamma(\alpha)}\int_0^s (s-\omega)^{\alpha-1}\int_0^{\omega}\int_0^C |\mathcal{K}(x,r)||\mathcal{H}(\mathcal{V}(r)) - \mathcal{H}(\mathcal{W}(r))|drdxd\omega \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\omega)^{\alpha-1}|\psi(\omega)|d\omega.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\mathcal{V}(s) - \mathcal{W}(s)\| &\le \|v_0 - w_0\| + \|v_1 - w_1\|C + \frac{\rho\, L\mathcal{K}^*\, C^{\alpha+2}}{\Gamma(\alpha+2)}\|\mathcal{V}(s) - \mathcal{W}(s)\| + \frac{C^{\alpha}\varepsilon}{\Gamma(\alpha+1)}, \\ &\le \|v_0 - w_0\| + \|v_1 - w_1\|C + \mathcal{R}\|\mathcal{V}(s) - \mathcal{W}(s)\| + \frac{C^{\alpha}\varepsilon}{\Gamma(\alpha+1)}, \\ &\le \frac{1}{1-\mathcal{R}}\left(1 + C + \frac{C^{\alpha}}{\Gamma(\alpha+1)}\right)\varepsilon.\end{aligned}$$

Let

$$\zeta_{\mathcal{H},\mathcal{K},\varphi} = \frac{1}{1-\mathcal{R}}\left(1 + C + \frac{C^{\alpha}}{\Gamma(\alpha+1)}\right).$$

Hence,

$$\|\mathcal{V}(s) - \mathcal{W}(s)\| \le \zeta_{\mathcal{H},\mathcal{K},\varphi}\,\varepsilon.$$

By Definition 3.1, the IVP (1) is H.U. stability. □

4 LADM Technique

In this section, we describe the application of LADM in solving Equation (1). This method is considered a simple iterative technique that combines the Laplace transform method and the Adomian decomposition method. The advantage of the method depends on its ability to provide an analytical solution for the complexity of non-linear FIDEs. Moreover, it yields a good approximation to the exact solution with low-cost computation and demonstrates the fast convergence of solutions. Note that the function $\mathcal{H}(\mathcal{W}(s))$ can be written as $\mathcal{H}(\mathcal{W}(s)) = R_{\mathcal{W}}(s) + N_{\mathcal{W}}(s)$, where $R_{\mathcal{W}}(s)$ is the linear part, and $N_{\mathcal{W}}(s)$ is the non-linear part. The second part of Equation (1) can be expressed as

$$\int_0^s \int_0^C \mathcal{K}(x,v)\mathcal{H}(\mathcal{W}(v))dvdx = \int_0^s \int_0^C \mathcal{K}(x,v)[R_{\mathcal{W}}(v) + N_{\mathcal{W}}(v)]dvdx. \tag{29}$$

Apply the Laplace transformation operator $\mathcal{L}$, where $\mathcal{L}(\mathcal{W}(s)) = W(\tau)$, and Lemma 2.1 on (1) and (29), gives

$$\tau^{\alpha}W(\tau) - \tau^{\alpha-1}\mathcal{W}(0) - \tau^{\alpha-2}\mathcal{W}'(0) = \mathcal{L}(\varphi(t)) + \rho\,\mathcal{L}\left(\int_0^s \int_0^C \mathcal{K}(x,v)[R_{\mathcal{W}}(v) + N_{\mathcal{W}}(v)]dvdx\right),$$

which reduces to

$$W(\tau) = \frac{w_0}{\tau} + \frac{w_1}{\tau^2} + \frac{1}{\tau^{\alpha}}\mathcal{L}(\varphi(s)) + \frac{\rho}{\tau^{\alpha}}\,\mathcal{L}\left(\int_0^s \int_0^C \mathcal{K}(x,v)[R_{\mathcal{W}}(v) + N_{\mathcal{W}}(v)]dvdx\right). \tag{30}$$

Consider the unknown solution $\mathcal{W}(s)$ as

$$\mathcal{W}(s) = \sum_{i=0}^{\infty} \mathcal{W}_i(s), \tag{31}$$

where $\mathcal{W}_i(s), i = 0, 1, 2, \ldots$ are evaluated recursively. The non-linear term $N_{\mathcal{W}}(s)$ is decomposed into the Adomian polynomial P_n in the form

$$N_{\mathcal{W}}(s) = \sum_{n=0}^{\infty} P_n(s), \tag{32}$$

where $P_n, n = 0, 1, 2, \ldots$ is defined by

$$P_n(s) = \frac{1}{n!}\frac{d^n}{d\lambda^n} N_{\mathcal{W}}\left(\sum_{i=0}^{n} \lambda^i \mathcal{W}_i(s)\right)\Bigg|_{\lambda=0}. \tag{33}$$

Also, the linear part can be written as

$$R_{\mathcal{W}}(s) = R_{\mathcal{W}}\left(\sum_{i=0}^{\infty} \mathcal{W}_i(s)\right). \tag{34}$$

From (30), we have

$$\begin{aligned}\mathcal{L}\left(\sum_{i=0}^{\infty} \mathcal{W}_i(s)\right) &= \frac{w_0}{\tau} + \frac{w_1}{\tau^2} + \frac{1}{\tau^{\alpha}}\mathcal{L}(\varphi(s)) \\ &\quad + \frac{\rho}{\tau^{\alpha}}\mathcal{L}\Bigg(\int_0^s \int_0^C \mathcal{K}(x,v)\left[R_{\mathcal{W}}\left(\sum_{i=0}^{\infty} \mathcal{W}_i(v)\right) + \sum_{i=0}^{\infty} P_i(v)\right] dvdx\Bigg).\end{aligned} \tag{35}$$

Let $\mathcal{W}_0$ be defined as all terms out of the integral sign, and the components $\mathcal{W}_i, i = 1, 2, \dots$ of the unknown function $\mathcal{W}(s)$ are evaluated recursively as follows:

$$\mathcal{L}(\mathcal{W}_0(s)) = \frac{w_0}{\tau} + \frac{w_1}{\tau^2} + \frac{1}{\tau^\alpha}\mathcal{L}(\varphi(s)), \tag{36}$$

and

$$\mathcal{L}\left(\mathcal{W}_{i+1}(s)\right) = \frac{\rho}{\tau^\alpha}\mathcal{L}\Big(\int_0^s \int_0^C \mathcal{K}(x,v)\left[R_{W_i}(v) + P_i(v)\right] dvdx\Big), \quad i = 0, 1, 2, \dots. \tag{37}$$

Now, apply the inverse Laplace transform to Equation (36), and the definition of Adomian polynomial in Equation (33) to find $P_0(s)$. Next, apply the recursive relation in Equation (37) to compute $\mathcal{W}_1(s)$. Completing in the same process, evaluating $P_1(s), P_2(s)\dots$ and $\mathcal{W}_2(s), \mathcal{W}_3(s)\dots$ gives

$$\mathcal{W}_0(s) = \mathcal{L}^{-1}\left(\frac{w_0}{\tau} + \frac{w_1}{\tau^2} + \frac{1}{\tau^\alpha}\mathcal{L}(\varphi(s))\right), \tag{38}$$

$$\mathcal{W}_{i+1}(s) = \mathcal{L}^{-1}\left(\frac{\rho}{\tau^\alpha}\mathcal{L}\Big(\int_0^s \int_0^C \mathcal{K}(x,v)\left[R_{W_i}(v) + P_i(v)\right] dvdx\Big)\right), \quad i = 0, 1, 2, \dots. \tag{39}$$

Finally, we obtain the series solution of $\mathcal{W}(s)$ which is expressed by Equation (31). In this work, we use the Mathematica and Matlab programs to obtain the numerical results. The convergence of LADM in Hilbert space was proved by [11], and in Banach space by [9, 5, 20].

5 Illustrative Examples

The following illustrative examples are provided to support the above developed theorems, and to show the effectiveness of the suggested method.

Example 5.1. *Consider the following non-linear FIDEs*

$$\begin{cases} {}^cD^\alpha_{0^+}\mathcal{W}(s) = -\frac{25}{504}s^2 + \frac{749}{360}s + \int_0^s \int_0^1 (x-v)(\mathcal{W}(v))^2 - \mathcal{W}(v))dvdx, \\ \mathcal{W}(0) = 1, \quad \mathcal{W}'(0) = 0, \quad 1 < \alpha \leqslant 2, \quad s \in [0,1]. \end{cases} \tag{40}$$

The exact solution is $\mathcal{W}(s) = \frac{1}{3}s^3 + 1.$

Solution: Using Theorem 3.1, (40) has at least one solution, since $\mathcal{H}(\mathcal{W}(s)), \varphi(s)$, and $\mathcal{K}(x,v) = (x-v)$ are continuous functions satisfying the conditions (A1)-(A3), such that $L = 1$, $\mathcal{K}^* = 1$, $\|\varphi(s)\| = 2.13016$, and $\mathcal{H}(\mathcal{W}(s))$ is bounded on $[0,1]$ with $z = 2$. Moreover, (40) has a unique solution and satisfies Theorem 3.2, since for each value of $\alpha \in (1,2]$, there exist a constant $\mathcal{R} \in [0.166667, 0.5]$. Furthermore, using Theorem 3.3, if we choose $\zeta \in [0.824361, 2.71828)$, then (40) is H.U. stability.

Applying the LADM as in Section 4, using (30)-(39), we obtain

$$\mathcal{W}_0(s) = \mathcal{L}^{-1}\left(\frac{1}{\tau^\alpha}\mathcal{L}\left(\frac{-25}{504}s^2 + \frac{749}{360}s\right) + \frac{1}{\tau}\right),$$

$$\mathcal{W}_{i+1}(s) = \mathcal{L}^{-1}\left(\frac{1}{\tau^{\alpha}}\mathcal{L}\left(\int_0^s \int_0^1 (x-v)\left[P_i(v) - \mathcal{W}_i(v)\right] dv dx\right)\right), \quad i = 0, 1, 2, ...,$$

where $P_n(s) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{n}\lambda^i \mathcal{W}_i(s)\right)^2\Big|_{\lambda=0}$. Then, the approximate solution can be expressed as a truncation series $\mathcal{W}(s) = \sum_{i=0}^{n}\mathcal{W}_i(s)$. Table 1 shows the comparison of the approximate solutions of LADM, Adomian decomposition method (ADM) (we use the same technique of ADM [39]), and homotopy analysis method (HAM) [16], for $N = n = 10$, $\alpha = 1.5$. It can be seen that LADM gives more accurate results. In Table 2, for the integer order $\alpha = 2$, it is observed that LADM provides more accurate results as compared to HAM and ADM.

Table 1: Comparison between the approximate solutions of LADM, ADM, and HAM at $N = 10$ and $\alpha = 1.5$ for Example 5.1.

s	Exact solution	Approximate solution		
		LDAM	ADM	HAM [16]
0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	1.000333333	1.0016633590	1.0018141266	1.0016059061
0.2	1.002666667	1.0094599621	1.0102811890	1.0099757479
0.3	1.009000000	1.0262079792	1.0283838147	1.0282561621
0.4	1.021333333	1.0540859024	1.0583734561	1.0585577810
0.5	1.041666667	1.0949840452	1.1021613866	1.1026362642
0.6	1.072000000	1.1506202268	1.1614487831	1.1620897325
0.7	1.114333333	1.2225962736	1.2377908721	1.2384279470
0.8	1.170666667	1.3124306243	1.3326342999	1.3330973985
0.9	1.243000000	1.4215791127	1.4473411780	1.4474901221
1	1.333333333	1.5514491321	1.5832056304	1.5829467357

Table 2: Absolute errors for LADM and HAM for $N = 10$ and $\alpha = 2$ for Example 5.1.

s	Absolute error	
	LADM	HAM [16]
0.1	0.0000000000	3.9968×10^{-14}
0.2	0.0000000000	3.0997×10^{-13}
0.3	0.0000000000	9.9987×10^{-13}
0.4	0.0000000000	2.3000×10^{-12}
0.5	2.22045×10^{-16}	4.3401×10^{-12}
0.6	2.22045×10^{-16}	7.2400×10^{-12}
0.7	0.0000000000	1.1070×10^{-11}
0.8	2.22045×10^{-16}	1.5900×10^{-11}
0.9	4.44089×10^{-16}	2.1700×10^{-11}
1	4.44089×10^{-16}	2.8588×10^{-11}

Example 5.2. *Consider the following non-linear FIDEs*

$$\begin{cases} {}^cD_{0^+}^{\alpha}\mathcal{W}(s) = \sqrt{s} + \displaystyle\int_0^s \int_0^1 xv(\mathcal{W}(v)^2 + \mathcal{W}(v))dv dx, \\ \mathcal{W}(0) = 0, \quad \mathcal{W}'(0) = 0, \quad 1 < \alpha \leqslant 2, \quad s \in [0, 1]. \end{cases} \tag{41}$$

The exact solution is $\mathcal{W}(s) = 1 + s^2/2 + s^3/6 + 0.053s^4$.

Solution: Using Theorem 3.1, (41) has at least one solution, since $\mathcal{H}(\mathcal{W}(s)), \varphi(s)$, and $\mathcal{K}(x, v)$ are continuous functions satisfying the conditions (A1)-(A3), such that $L = \|\mathcal{W}(s_1)+\mathcal{W}(s_2)\|^2 \leq 2M^2$, where $\|\mathcal{W}(s)\| \leq M$, M positive constant, and for $s, s_1, s_2 \in [0, 1]$. Also, $\mathcal{K}^* = 1$, $\|\varphi(s)\| = 1$, and $\mathcal{H}(\mathcal{W}(s))$ is bounded on $[0, 1]$ with $z = L|s_1 - s_0|$, for some $s_1, s_0 \in [0, 1]$. Moreover, (41) has a unique solution and satisfies Theorem 3.2, since for each value of $\alpha \in (1, 2]$, there exist a constant $\mathcal{R} = \dfrac{2M^2}{\Gamma(\alpha+2)} < 1$. Furthermore, using Theorem 3.3, if we choose $\zeta \in [0.824361, 2.71828)$, then (41) is H.U. stability.

Applying the LADM as in Section 4, using (30)-(39), we obtain

$$\mathcal{W}_0(s) = \mathcal{L}^{-1}\left(\frac{1}{\tau^\alpha}\mathcal{L}\left(\sqrt{s}\right)\right),$$

$$\mathcal{W}_{i+1}(s) = \mathcal{L}^{-1}\left(\frac{1}{\tau^\alpha}\mathcal{L}\left(\int_0^s\int_0^1 (xv)\left[P_i(v) + \mathcal{W}_i(v)\right] dv dx\right)\right), \ i = 0, 1, 2, ...,$$

where $P_n(s) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^n \lambda^i \mathcal{W}_i(s)\right)^2\Big|_{\lambda=0}$. The approximate solution is expressed as a truncation series $\mathcal{W}(s) = \sum_{i=0}^n \mathcal{W}_i(s)$. Table 3 indicates that as α approaches 2, the approximate solution approaches the exact solution.

Table 3: The approximate solutions of LADM with $N = 10$ and different values of α for Example 5.2.

s	Exact	LADM			
	solution	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 1.8$	$\alpha = 2$
0.1	1.0237897099	1.3569879309	1.1000165642	1.0429610557	1.0237897099
0.2	1.0673057489	1.5059313665	1.2001874032	1.1058262807	1.0673057489
0.3	1.1237202080	1.6224426329	1.3007746338	1.1794299508	1.1237202080
0.4	1.1906620121	1.7240885435	1.4021202254	1.2611842983	1.1906620121
0.5	1.2668292930	1.8182730316	1.5046298518	1.3498231505	1.2668292930
0.6	1.3514149115	1.9092700221	1.6087639812	1.4446532848	1.3514149115
0.7	1.4438995165	2.0000157079	1.7150319463	1.5453011436	1.4438995165
0.8	1.5439553073	2.0927641855	1.8239876128	1.6515998079	1.5439553073
0.9	1.6513942138	2.1893800318	1.9362260391	1.7635299038	1.6513942138
1	1.7661371433	2.2914869341	2.0523807938	1.8811852664	1.7661371433

Example 5.3. *Consider the following non-linear FIDEs*

$$\begin{cases} {}^cD^\alpha_{0^+}\mathcal{W}(s) = s + 1 + \displaystyle\int_0^s\int_0^1 xv\mathcal{W}(v))^3 dv dx, \\ \mathcal{W}(0) = 1, \quad \mathcal{W}'(0) = 0, \quad 1 < \alpha \leqslant 2, \quad s \in [0, 1]. \end{cases} \tag{42}$$

The exact solution is $\mathcal{W}(s) = 1 + s^2/2 + s^3/6 + 0.053s^4$.

Solution: Using Theorem 3.1, (42) has at least one solution, since $\mathcal{H}(\mathcal{W}(s)), \varphi(s)$, and $\mathcal{K}(x,v)$ are continuous functions satisfying the conditions (A1)-(A3), such that $L = 1, \mathcal{K}^* = 1, \|\varphi(s)\| = 1$, and $\mathcal{H}(\mathcal{W}(s))$ is bounded on $[0,1]$ with $z = 1$. Moreover, for each value of $\alpha \in (1,2]$, there exist a constant $\mathcal{R} = \dfrac{1}{\Gamma(\alpha+2)} < 1$, such that (42) satisfies Theorem 3.2. Furthermore, using Theorem 3.3, if we choose $\zeta = \frac{1}{\Gamma(\alpha+1)} \times \exp(\frac{1}{\Gamma(\alpha+1)}) \in [0.824361, 2.71828)$, then (42) is H.U. stability.

Applying the LADM as in Section 4, using (30)-(39), we obtain

$$\mathcal{W}_0(s) = \mathcal{L}^{-1}\left(\frac{1}{\tau^\alpha}\mathcal{L}\left(s+1\right) + \frac{1}{\tau}\right),$$

$$\mathcal{W}_{i+1}(s) = \mathcal{L}^{-1}\left(\frac{1}{\tau^\alpha}\mathcal{L}\left(\int_0^s \int_0^1 xvP_i(v)dvdx\right)\right), \ i = 0,1,2,...,$$

where $P_n(s) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^n \lambda^i \mathcal{W}_i(s)\right)^3\Big|_{\lambda=0}$. The approximate solution is expressed as a truncation series $\mathcal{W}(s) = \sum_{i=0}^n \mathcal{W}_i(s)$. Figure 1 exhibits the approximate solutions of LADM for different values of α, and notes that when $\alpha = 2$, the approximate solution of Equation (42) coincides with the exact solution.

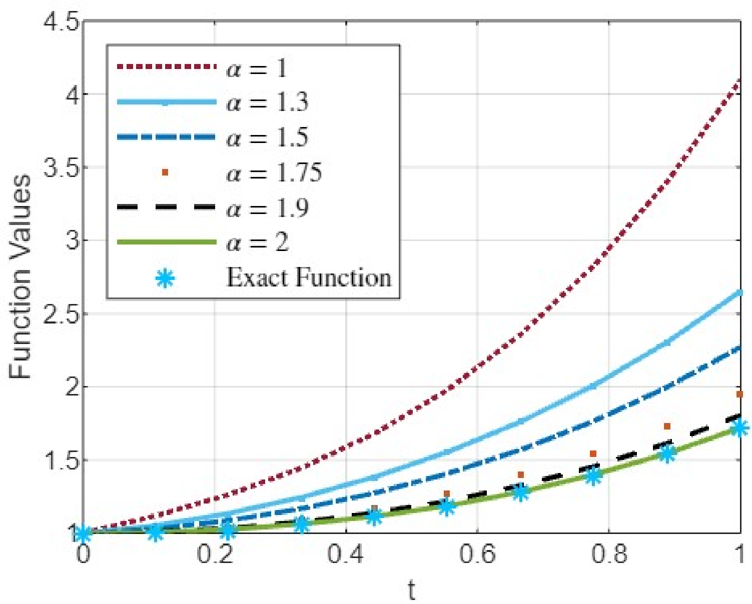

Figure 1: Approximate solution for LADM at different values of α and $N = 6$ for Example 5.3.

Example 5.4. *Consider the following linear FIDEs*

$$\begin{cases} {}^cD^\alpha_{0^+}\mathcal{W}(s) = -15s + \displaystyle\int_0^s \int_0^1 xv\,\mathcal{W}(v)dvdx, \\ \mathcal{W}(0) = 1, \quad \mathcal{W}'(0) = 0, \quad 1 < \alpha \leqslant 2, \quad s \in [0,1]. \end{cases} \tag{43}$$

The exact solution is $\mathcal{W}(s) = 1 - \dfrac{5}{2}s^3$.

Solution: Using Theorem 3.1, (43) has at least one solution, since $\mathcal{H}(\mathcal{W}(s)) = \mathcal{W}(s)$, $\varphi(s) = -15s$, and $\mathcal{K}(x,v) = xv$ are continuous functions satisfies the conditions (A1)-(A3), such that $L = 1$, $\mathcal{K}^* = 1$, $\|\varphi(s)\| = 15$, and $\mathcal{H}(\mathcal{W}(s))$ is bounded on $[0,1]$ with $z = 1$. Moreover, for $\rho = 1, C = 1$, there exist a constant $0.166667 \leq \mathcal{R} = \dfrac{1}{\Gamma(\alpha+2)} \leq 0.5$ depends on the different values of $\alpha \in (1,2]$, such that (42) satisfying Theorem 3.2. Furthermore, using Theorem 3.4, if we choose $\zeta > 0$, such that $\zeta \in [3,6)$, where $\alpha \in (1,2]$, then (43) is H.U. stability.

Applying the LADM as in Section 4, using (30)-(39), we obtain

$$\mathcal{W}_0(s) = \mathcal{L}^{-1}\left(\frac{1}{\tau} + \frac{1}{\tau^\alpha}\mathcal{L}(-15s)\right),$$

$$\mathcal{W}_{i+1}(s) = \mathcal{L}^{-1}\left(\frac{1}{\tau^\alpha}\mathcal{L}\left(\int_0^s\int_0^1 xv\mathcal{W}_i(v)dvdx\right)\right), i = 0,1,2,...,$$

where $P_n(s) = 0$. The approximate solution is expressed as a truncation series $\mathcal{W}(s) = \sum_{i=0}^{n}\mathcal{W}_i(s)$. Figure 2 exhibits the approximate solutions of LADM for different values of α and notes that when $\alpha = 2$, the approximate solution of LADM coincides with the exact solution.

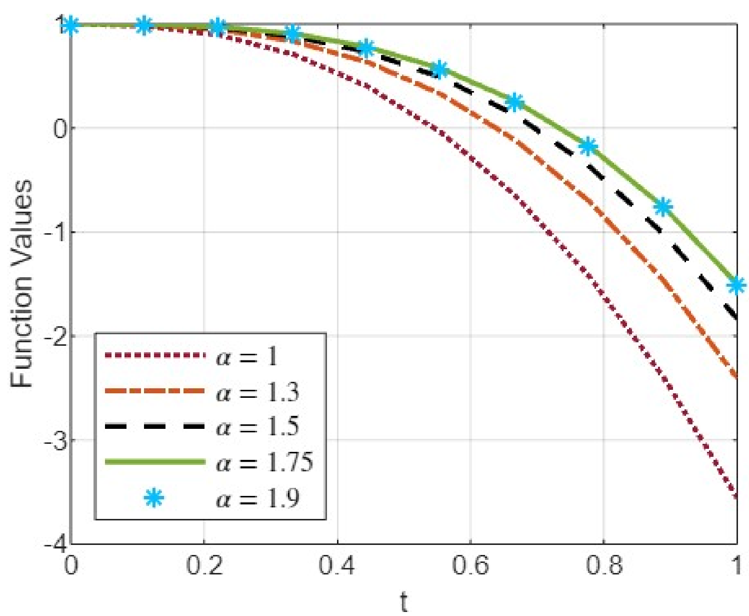

Figure 2: Approximate solution for LADM at some values of α and $N = 10$ for Example 5.4.

6 Conclusion

This paper successfully proved the existence and uniqueness of solutions for FVFIDEs in the Caputo sense in Banach space, employing the fixed-point theorems under some conditions. The stability of solutions was constructed with H.U. stability using various techniques. The LADM was applied to solve the given equation, and some numerical examples were provided to demonstrate the effectiveness of LADM, showcasing its superiority in approximating the solutions as compared to ADM and HAM. The provided examples also supported the constructed theorem.

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